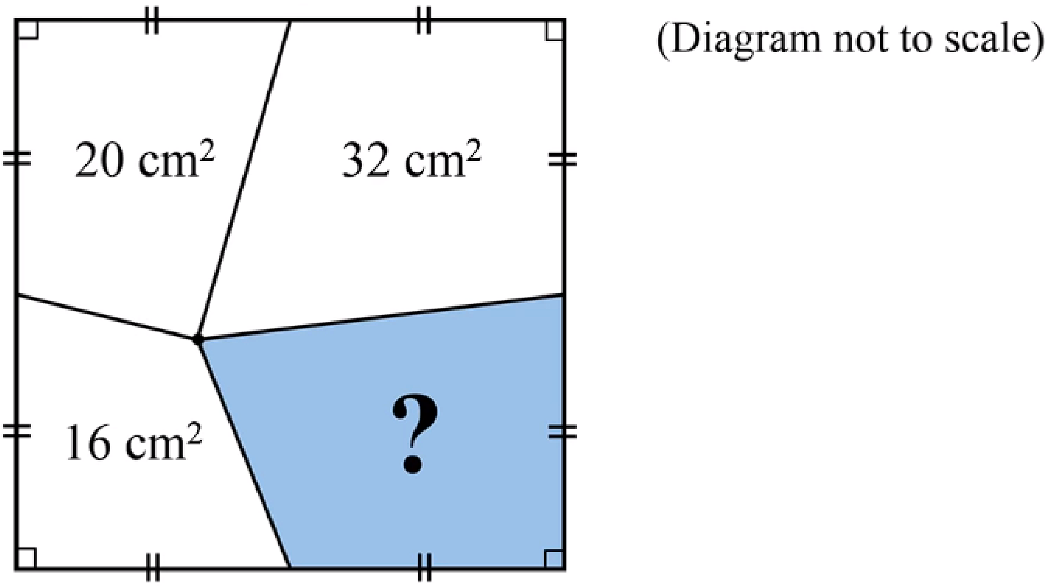
October Math Gems

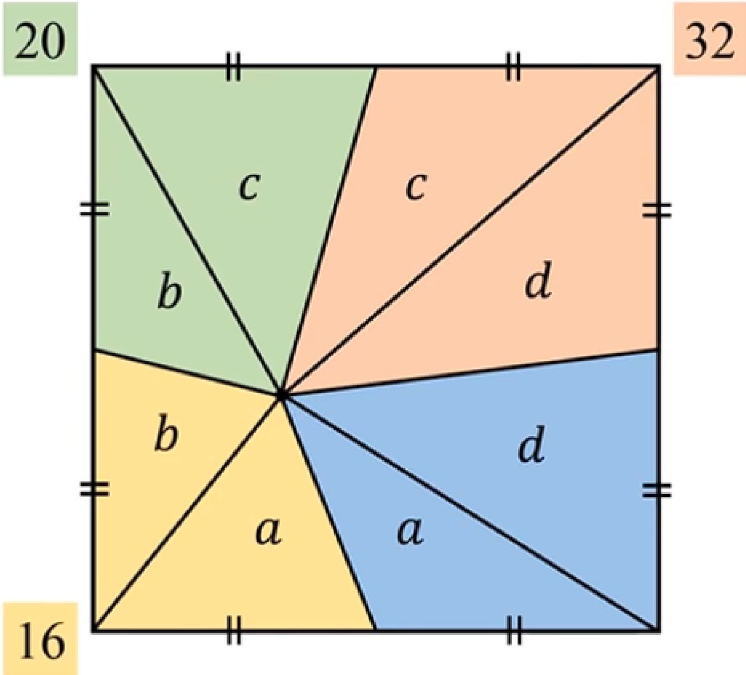
Problem of the week 34

§1 Problems

Problem 1.1. Find the total area of the following figure:



Solution. As we know, the area of any triangle is $\frac{1}{2}.B.h$, if you divided the following figure:



As you can see it's clear that every 2 triangles carrying the same letter have the same base and height.
since, a+b=16, b+c=20, and c+d= 32.
since the sum of opposite areas are equal.
$\therefore$ (a+b)+(c+d)=(b+c)+(a+d)
$\therefore$ 16+32=20+(a+d)
$\therefore$ (a+d)= 28
then, the total area of the figure=
20+16+32+28= $96cm^2$

$\square$

Problem 1.2. If 3X-Y=12,

$$\text{what is the value of } \frac{8^x}{2^y}$$

Solution. $\because 2^3 = 8$
$\therefore \frac{8^X}{2^Y} = \frac{2^{3.X}}{2^Y}$
$\because \frac{X^Y}{X^Z} = X^{Y-Z}$
$\therefore \frac{2^{3.X}}{2^Y} = 2^{3X-Y}$

$$= 2^{12} = 4096$$

$\square$

Problem 1.3.

$$(4+\sqrt{15})^x + (4-\sqrt{15})^x = 62$$

find the value of x.

Solution.

$$\because 4+\sqrt{15} \text{ and, } 4-\sqrt{15} \text{ are conjugates}$$

$$\therefore 4-\sqrt{15}=\frac{1}{4+\sqrt{15}}$$

then, let

$$Y=(4+\sqrt{15})^x$$

and

$$\frac{1}{y}=(4-\sqrt{15})^x$$

$$\therefore y+\frac{1}{y}=62$$

$$\therefore y^2+1=62y$$

$$\therefore y^2-62y+1=0$$

$$\therefore y=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{62\pm\sqrt{(-62)^2-4(1)(1)}}{2(1)}$$

$$=\frac{62\pm 16\sqrt{15}}{2}$$

$$=\frac{62}{2}\pm\frac{16\sqrt{15}}{2}$$

$$=31\pm 8\sqrt{15}$$

$$y=16+15\pm 2.4\sqrt{15}$$

=

$$4^2\pm 2.4\sqrt{15}+(\sqrt{15})^2$$

=

$$(4\pm\sqrt{15})^2$$

$$\therefore y=(4-\sqrt{15})^2 \text{ or } (4+\sqrt{15})^2$$

$$\because Y=(4+\sqrt{15})^x \text{ and } \frac{1}{y}=(4-\sqrt{15})^x$$

$$\therefore (4+\sqrt{15})^x=Y=(4+\sqrt{15})^2$$

$$\therefore x=2$$

$$\therefore (4+\sqrt{15})^x=(4-\sqrt{15})^2$$

$$\therefore (4+\sqrt{15})^x=\frac{1}{(4+\sqrt{15})^2}$$

$$\therefore (4+\sqrt{15})^x=(4+\sqrt{15})^{-2}$$

$$\therefore x=-2$$

$$\therefore x=\pm 2$$

□

Problem 1.4.

$$3^{x+2} = 9^{2x-3}$$

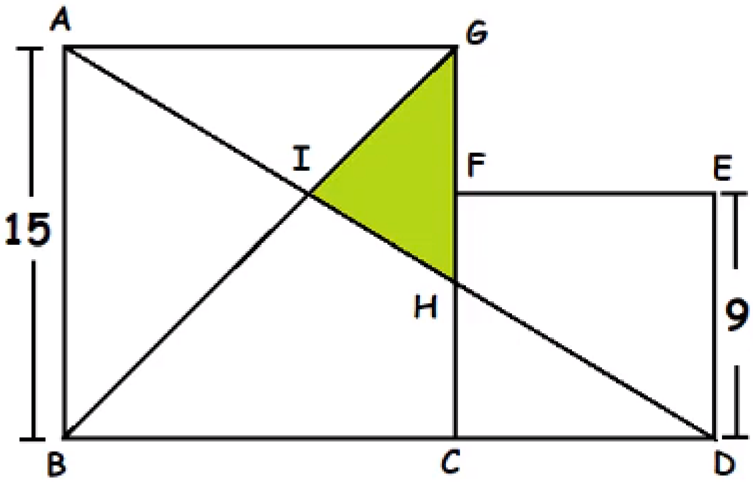
find the value of x.

Solution. $\because 3^2 = 9$
$\therefore 9^{2x-3} = 3^{2(2x-3)}$
Then, $3^{x+2} = 3^{2(2x-3)}$
$\because$ If $x^y = x^z$, then $y = z$
$\therefore x + 2 = 2(2x - 3)$
$\therefore x + 2 = 4x - 6$
$\therefore, 3x = 8$

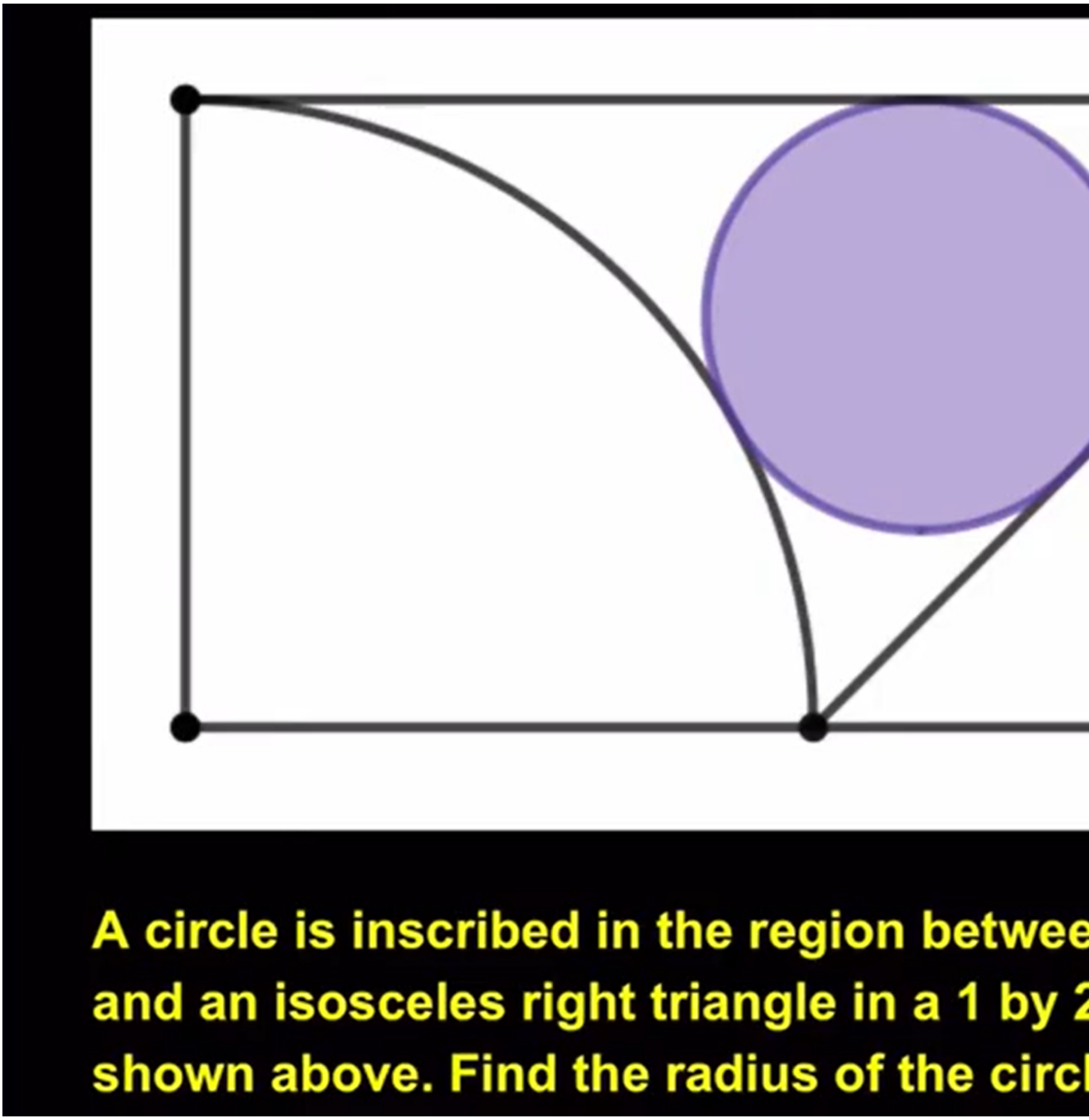
$$\therefore x = \frac{8}{3}$$

□

Problem 1.5. Find the area of the green part(where units are in cm).



Solution. $\frac{CD}{BD} = \frac{CH}{AB}$
$\therefore \frac{9}{24} = \frac{CH}{15}$
$\therefore CH = \frac{45}{8}$
$\therefore GH = 15 - \frac{45}{8} = \frac{75}{8}$
$\frac{GIH}{GIA} = \frac{IH}{IA} = \frac{GH}{GA}$ Draw a straight line from point p to h as shown in the following



$then, APHG = 15.\frac{75}{8}$
$AGH = \frac{15.75}{16} = GIH + GIA$
$\frac{GIH}{GIA} = \frac{IH}{IH+IA} = \frac{GH}{AG+GH}$
$\therefore \frac{GIH}{\frac{15.75}{16}} = \frac{\frac{75}{8}}{\frac{15.75}{16}}$

$$= \frac{5625}{208} cm^2$$

□

Problem 1.6.

$$\sqrt[3]{44X - 7} - 5 = 0$$

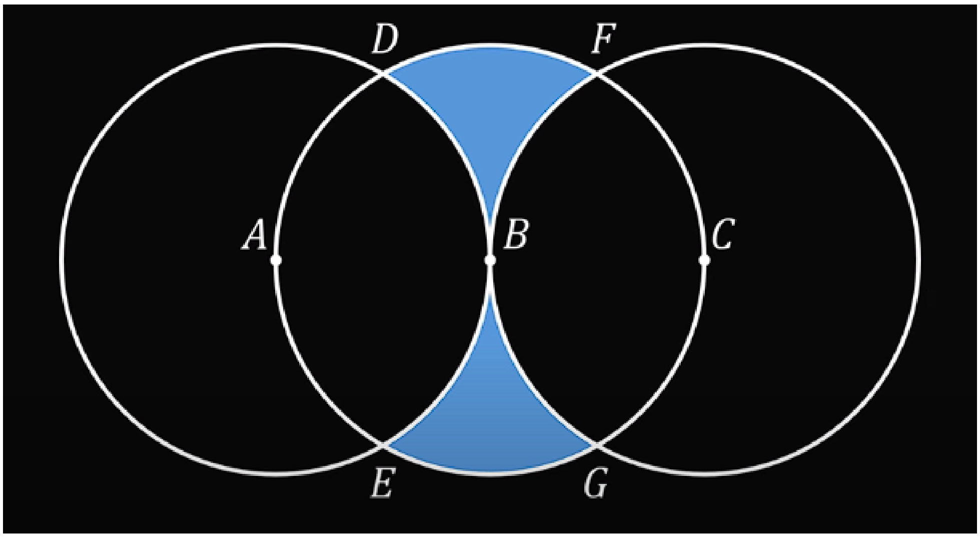
find the value of x.

Solution. $\sqrt[3]{44X-7}-5=0$
$\sqrt[3]{44X-7}=5$
44X-7=5^3
$44X=125+7$
X=$\frac{132}{44}$

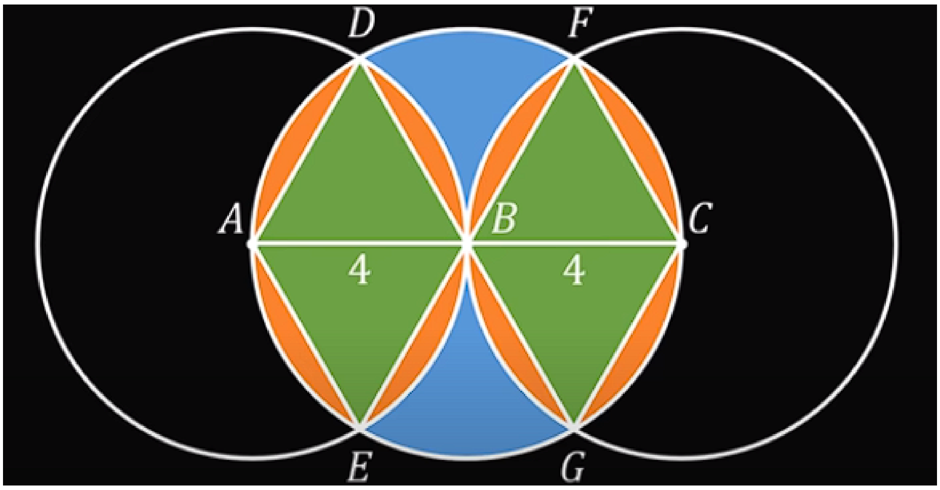
$$\therefore X=3$$

□

Problem 1.7. find the area of the blue part, where r=4.

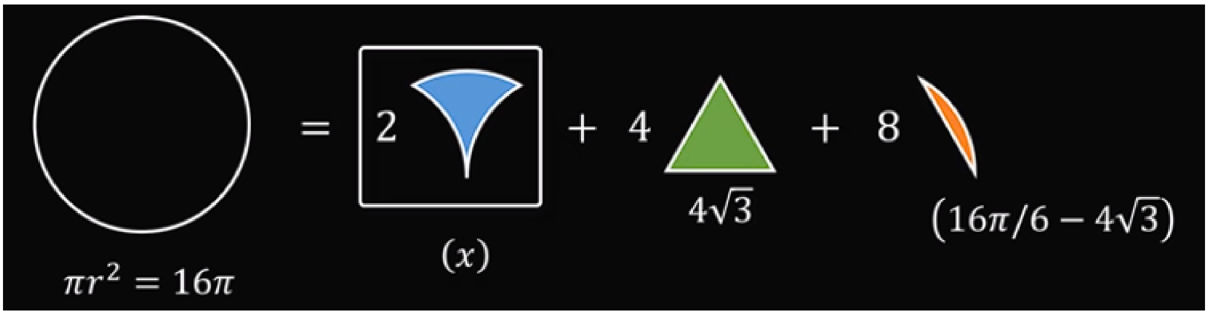


Solution. Firstly, connect each circle's center to other centers, intersection points, points of tangency, etc.
Secondly, break down areas into circular sectors and polygons.
After doing the previous, you must end up with this shape:



The area of the orange part is equal to the difference between the area of the sector and the area of the green triangle, as the following figure shows:

The area of the white sector is equal to: $\pi(4^2).\frac{60}{360} = \frac{16\pi}{6}$
The area of the green triangle is equal to $\frac{4^2\sqrt{3}}{4} = 4\sqrt{3}$.
$\therefore$ the area of the orange part is equal to $\frac{16\pi}{6} - 4\sqrt{3}$
Now, to calculate the area of the blue part we will use the fact that the middle circle area is equal to the sum of the areas of the blue, yellow, and green parts.
As we know the area of the circle is equal to πr^2
So, we will use the following equation in order to calculate the area of the blue part:



$\therefore 16\pi = x + 4(4\sqrt{3}) + 8(\frac{16\pi}{6} - 4\sqrt{3})$
$16\pi = x + 16\sqrt{3} + \frac{128\pi}{6} - 32\sqrt{3}$
$0 = x - 16\sqrt{3} + \frac{32\pi}{6}$

$$\therefore x = 16\sqrt{3} - \frac{16\pi}{3} \simeq 10.958$$

$\square$

Problem 1.8. The surface area of a human's lungs is equal to half of a tennis court ($2100\ ft^2$). How many square feet would the lungs of a 30-person baseball team cover?

Solution. $\frac{1050ft^2}{1lungs} = \frac{A}{30lungs}$

$$\therefore A = 31500$$

$\square$

Problem 1.9. At a shop in Times Square one "I LOVE NY" t-shirt is sold every 10 minutes for 19.95 dollars each. The shop opens from 9 am until 9 pm every day.
How many t-shirts are sold in a week?

Solution. The shop opens from 9 am to 9 pm, so it opens 12 hours a day. $\because$ 12h = 5040 min. $\therefore$ the shop opens 5040 min. each week. $\because$ there is a T-shirt sold every 10 min. $\therefore$ by making the ratio $\frac{1}{10min} = \frac{x}{5040}$, then the shop solds 504 shirts every week. $\square$

Problem 1.10.

$$(x+3)(x-5) = 5$$

Find all the solutions to the equation above.

Solution. $\because$(x+3)(x-5)=5
$\therefore x^2 - 2x - 15 = 5$
$\therefore x^2 - 2x - 20 = 0$
$\therefore$ by applying the rule $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$\therefore x = 1 \pm \sqrt{21}$$

□

Problem 1.11. The leaning tower of Pisa in Italy leans at an angle of 84.7 degrees. 171ft from the base of the tower, the angle of elevation to the top is 50 degrees. Find the height of the tower if it was sitting straight up.

Solution. draw the given thin like the following

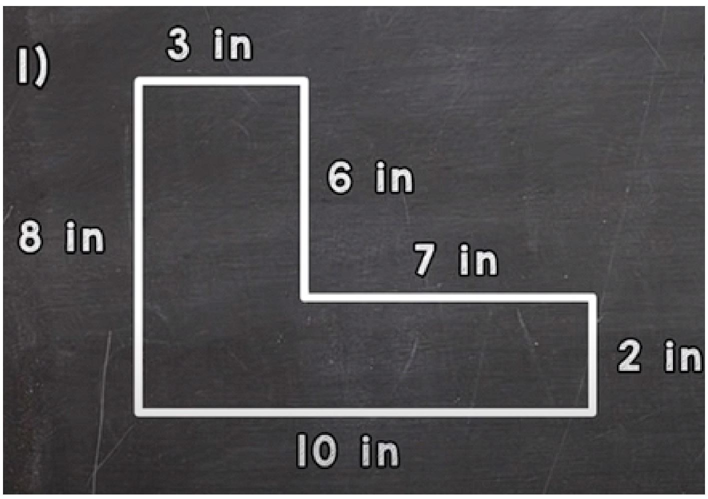
idwutn.PNG

let c be the height of the tower.
the measure of angle B is 180-(84.7+50)=45.3
then to calculate AB which is c, we will apply the sin law.
$\because \frac{b}{sin(B)} = \frac{c}{sin(C)}$
$\because \frac{171}{sin(45.3)} = \frac{c}{sin(50)}$

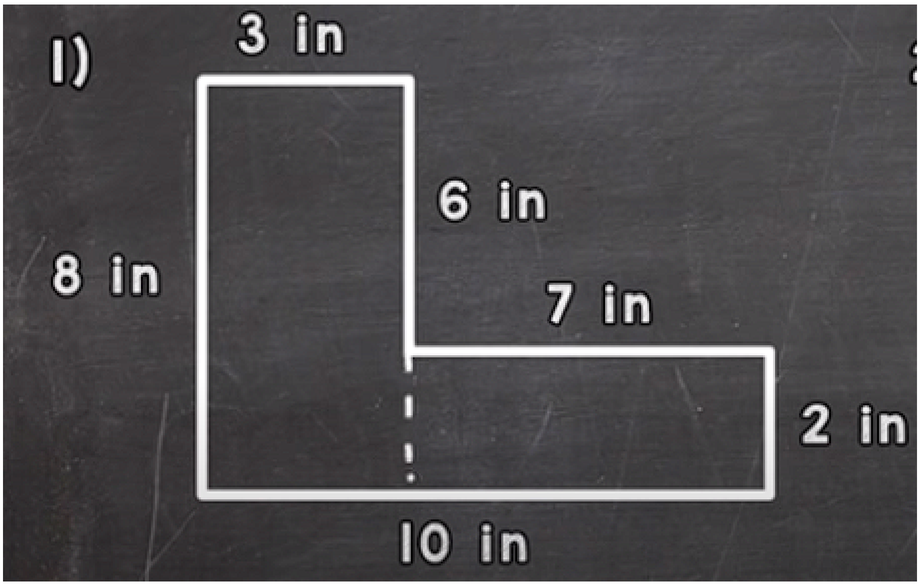
$$\therefore c = 184ft$$

□

Problem 1.12. Find the area of the following figure:



Solution. To calculate the area separate the figure into two rectangles.

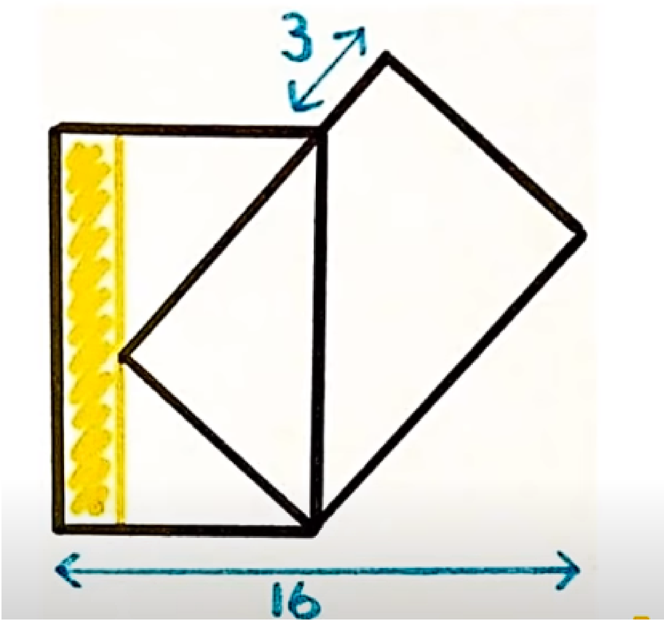


Calculate the area of the vertical and the horizontal ones separately, then add them.
The area of the vertical rectangle equals, $8.3 = 24in^2$
The area of the horizontal one equals, $7.2 = 14in^2$

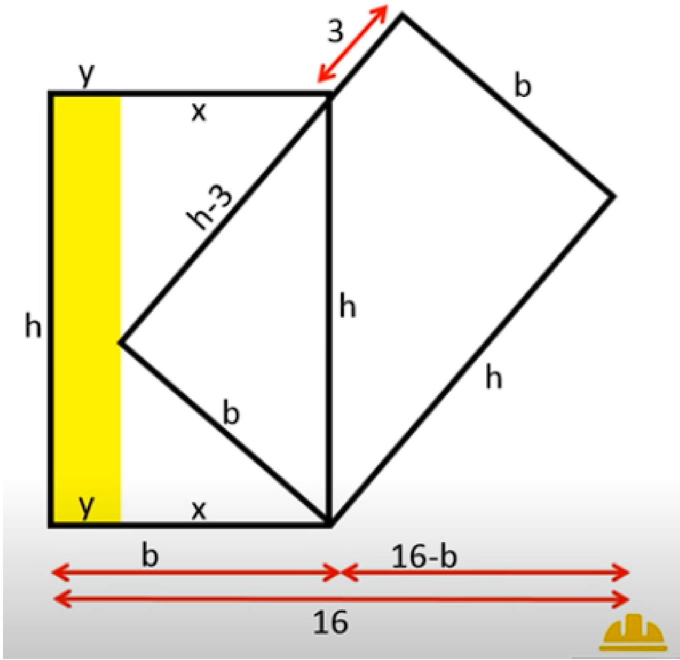
$$\therefore \text{ the total area equals } 24 + 14 = 38in^2$$

$\square$

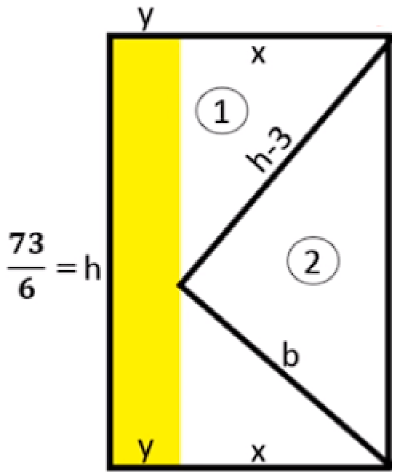
Problem 1.13. Find the area of the yellow part, knowing that the dimensions are in cm unit.



Solution. First, let's name each part on the figure



Now, the area of the shaded part can be determined as h.y (as it's clearly a rectangle) to calculate the value of h,
$h^2 = (h-3)^2 + b^2 = h^2 - 6h - 9 + 64$
$\therefore 0 = 64 - 6h - 9$
$\therefore h = \frac{73}{6} cm^2$
Now, to find y, number the triangles in the vertical rectangle like this



so, we get that, $\frac{x}{h-3} = \frac{b}{h}$
$\therefore \frac{x}{\frac{73}{6}-3} = \frac{8}{\frac{73}{6}}$
$\therefore x = \frac{144}{73} cm^2$
$\therefore y = 8 - \frac{144}{73} \simeq 1.973 cm^2$
$\therefore theareaoftheyellowpart = \frac{73}{6}.1.973$

$$= 24cm^2$$

□

Problem 1.14. Suppose A and B are positive real numbers for which $\log_A B = \log_B A$. If neither A nor B is 1 and $A \neq B$, find the Value of AB

Solution. $log_a b = log_b a$
$\frac{log(b)}{log(a)} = \frac{log(a)}{log(b)}$
$(log(a))^2 = (log(b))^2$
$(log(a))^2 - (log(b))^2 = 0$
$(log(a) - log(b)).(log(a) + log(b)) = 0$
$\therefore (log(a) + log(b)) = 0$
$log(a.b) = 0$
$\therefore 10^0 = ab$

$$\therefore ab = 1$$

□

Problem 1.15.

$$log_2(11 - 6x) = 2log_2(x - 1) + 3$$

find the value x.

Solution. $log_2(11 - 6x) = 2.log_2(x - 1) + 3$
$log_2(11 - 6x) - (log_2(x - 1))^2 = 3$
$log_2 \frac{11-6x}{(x-1)^2} = 3$
$\frac{11-6x}{(x-1)^2} = 2^3$
$11 - 6x = 8(x - 1)^2$
$11 - 6x = 8(x^2 - 2x + 1)$
$11 - 6x = 8x^2 - 16x + 8$
$\therefore 8x^2 - 10x - 3 = 0$
$\therefore x = \frac{-1}{4}$ or $\frac{3}{2}$
If you check the answer you will find that only $\frac{3}{2}$ is the value of x, as $x - 1 > 0$.

□

Problem 1.16. if f(x)= $x^2 + 1$ then what is (f.f)(x)?

Solution. $\because f(x) = x^2 + 1$
$\therefore f(f(f)) = (x^2 + 1)^2 + 1$
$= (X^2)^2 + 2x^2 + 1 + 1$

$$= x^4 + 2x^2 + 2$$

□

Problem 1.17. Solve $cos^2(\frac{x}{2}) = cos^2(x)$, on the interval $0 \leq x < 2\pi$.

Solution. $\because cos^2(\frac{x}{2}) = cos^2(x)$
$\therefore (cos(\frac{x}{2}))^2 = cos^2(x)$
$\therefore (\pm\sqrt{\frac{1+cos(x)}{2}})^2 = cos^2(x)$
$\therefore \frac{1+cos(x)}{2} = cos^2(x)$
$\therefore 1 + cos(x) = 2.cos^2(x)$
$\therefore 2.cos^2(x) - cos(x) - 1 = 0$

$\therefore (2cos(x)+1)(cos(x)-1)$
$\therefore cos(x) = \frac{-1}{2}$ or $cos(x) = 1$

$$\therefore x = 0\ ,\ \frac{4\pi}{3} \text{ and, } \frac{2\pi}{3}$$

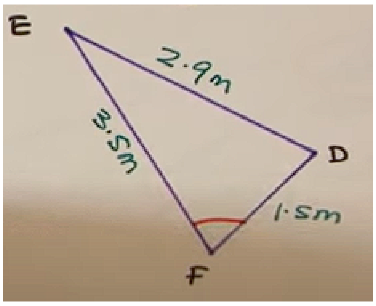
□

Problem 1.18. Determine the period of

$$y = 8 + 6cos(\frac{2\pi x}{15})$$

Solution. $\because$ the period of cos is $\frac{2\pi}{b}$
$\therefore \frac{2\pi}{b} = \frac{2\pi}{\frac{2\pi}{15}}$
then, multiply both sides by $\frac{15}{2\pi}$
so, the final answer will be 15. □

Problem 1.19. Find angle F



Solution. using the cos law
$\therefore cos(f) = \frac{d^2+e^2-f^2}{2(1.5)(3.5)}$
$\therefore cos(f) = \frac{3.5^2+1.5^2-2.9^2}{2(1.5)(3.5)}$
$\therefore cos(f) = 0.58$

$$\therefore F \simeq 54.5 \text{ degrees}$$

□

Problem 1.20. Determine the range of the function

$$f(x) = 4 - 2sin(3x)$$

Solution. $\because f(x) = 4 - 2sin(3x)$
$\therefore f(x) = -2sin(3x) + 4$
$\therefore$ the function has an amplitude of 2
But, the +4 at the end of the function means that it will be shifted up by 4.

$$\therefore \text{ the range is } [2,6]$$

□